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A MODEL FOR CASH BALANCE MANAGEMENT*†

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A model for minimizing the average cash balance subject to a constraint on the probability of stock-out is presented. The cash balance is described as an inventory process that changes because of deterministic and stochastic events. Recursive systems of equations are given to find (1) the distribution function of the cash level at any time and (2) the probability that all demands during some time interval are met. Then we examine the problem: minimize the expected time average cash balance subject to the condition that the probability that all demands are satisfied is at least some given number. It is shown that the optimal policy has a very simple form, which can be expressed verbally as, "never have any more cash on hand than is necessary to satisfy the constraint."

In [2], Girgis formulated the problem of selecting a cash level in anticipation of future net expenses as a single product multiperiod inventory system. The essential features of her model are: (i) a holding cost when there is a positive cash level, (ii) a shortage cost when there is a negative cash level, (iii) a fixed charge for choosing to change the cash level, (iv) the ability to decrease or increase the cash balance in every period, and (v) net expenses in each period are independent and identically distributed random variables. In this paper we will retain features (i) and (iv), replace (iv) by a lower bound on the probability of stockout, and relax (v) by not requiring the random variables to be identically distributed. We shall eliminate (iii) because, in large firms, the only marginal costs associated with obtaining cash are the flotation costs when securities are issued, and these are relatively small.

Following Linhart [4], we shall refer to the stock of cash as the *pool of funds*, and we consider the following mode of operation: The level of the pool can be increased periodically, once per month say, by selling stocks and/or bonds; if during a period there is a demand for cash that cannot be met by the pool, a bank loan at interest will be made for the balance. The pool manager wants to keep the average level of the pool as small as possible, but he does not want to borrow from the banks too often; this last goal is expressed by requiring that the probability that the pool fails to meet a demand is less than a specified number.

This type of inventory model has not received wide attention in the literature. Most authors e.g., [1], [2], and [5] have analyzed the global problem of minimizing a measure of the total cost of the process, the stock-out probability is found after the optimal policy is derived. (An exception to this is [3].) In the problem at hand, the author's opinion is that (1) the "cost" of having to borrow money from a bank cannot be computed in practice, and (2) a financial manager may view his responsibility as achieving a given level of financial stability at minimum cost. These considerations lead to the model described below.

In the next two sections we will present a model of the pool that leads to expressions for the average level of the pool and the probability of an emergency bank loan in

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terms of some financial control variables. §§ three and four contain the explicit solutions to the single-period and multiperiod models respectively, and the form of the optimal policy is shown in § five.

1. Problem Formulation

Even though the money in the pool is usually in the form of short-term securities, we can give it the same liquidity as a cash deposit because these notes are traded in a market. In order to keep the remaining presentation as straightforward as possible, the fact that money in the pool earns interest will be ignored. Later on it will be apparent how to introduce the effect of interest, and that ignoring it makes no essential difference in our results.

Let time zero be a conveniently chosen reference point, and let us measure time in days. Following Linhart [4], we shall assume that one day, three types of events may occur that will change the level of the pool, and these three types are classified as financial, calendar and random. A financial event is a stock issue, bond issue or planned bank loan. Let f_n denote the size of the financial event on day n; f_n is our decision variable. The calendar events are nonrandom payments (or receipts) that occur on specific dates, such as state and federal tax payments, interest payments on debt, dividend payments, and dividend payments from subsidaries. The size of the deposit in the pool, on n day from these sources will be denoted by d_n, and $d_n < 0$ represents a withdrawal from the pool. We observe that d_n may depend on f_m, $m \leqq n$, e.g., selling a bond in January results in an interest payment in July. Thus, we regard d_n as a known function of earlier decision variables (at least for the period under study). The random event on day n, denoted X_n, is most accurately described by introducing X'_n = the revenues received by the company on day n and X''_n = the expenditures of the company on day n, where $X'_n > 0$ and $X''_n > 0$. Then, $X_n \triangleq X'_n - X''_n$, i.e., X_n is the net revenue produced by the company on day n.

We can proceed in two ways, either (i) the demands will be back ordered until money is made available, or (ii) the demands will be met on the same day by making an emergency loan equal to the deficiency at the end of day n. We shall adopt the latter alternative for our model, but some results for the former will necessarily be obtained. When an emergency loan is made, the interest and principle are repaid at the end of the month. We assume that the loan interest rate is independent of the size of the loan and the number of loans made in the past.

The decision problem is to minimize the interest costs due to holding money in the pool and making emergency loans, subject to the constraint that the probability of requiring an emergency loan during each period of time (a month say) be at most some given number (less than one). Only finite horizon problems are considered.

2. The Model for a Single-Period Problem

Define the *inventory level* of the pool by

$$I_n = I_{n-1} + f_n + d_n + X_n, \qquad n = 1, 2, \cdots, \tag{1}$$

with $I_0 = I$ = the known initial cash balance; we observe that I_n may be negative. Since the level of the pool will always be kept nonnegative by emergency loans, if necessary, we define the *cash balance* on day n as P_n, where $P_0 = I_0$ and

$$P_n = \max\,[0, P_{n-1}] + f_n + d_n + X_n, \qquad n = 1, 2, \cdots. \tag{2}$$

If $P_n < 0$, an emergency loan of size $-P_n$ is required to satisfy the demand on the pool, and the random variable E_n defined by

$$E_n = -P_n \quad \text{if} \quad P_n < 0, \qquad (3)$$
$$\quad = 0 \qquad \text{if} \quad P_n \geqq 0,$$

will represent the size of the emergency loan on day n to keep the pool nonnegative.

We will denote by N' the set of days in the planning horizon, $N' = \{1, 2, \cdots, N\}$. The constraint of the model, that the probability of no emergency loan during N' be at most a given number, $1 - \beta$ say $(0 < \beta < 1)$, can be written as

$$B_N^*(I) = \Pr\{E_1 = 0, \cdots, E_N = 0 \mid I_0 = I\} \geqq \beta. \qquad (4)$$

Assume that on day zero we can instantaneously add f more dollars, $-\infty < f < \infty$ to the cash balance, and that during the next N days no more financial inputs are allowed. If an emergency loan is made during the period, it will be repaid after the end of the period, so we need not consider the cash outflows from the repayments of emergency loans; the same assumption applies to repayments for the financial input f. The one-period optimization problem is to find that value of f, say f^*, that minimizes the time-average expected level of the cash balance during the planning period, denoted by $H_N(f; I)$. Letting $E(P_n; I + f)$ be the expected value of P_n when $P_0 = I + f$, we have

$$H_N(f; I) = N^{-1} \sum_{n=1}^{N} E(P_n; I + f) \qquad (5)$$

as the function to be minimized. To insure that a value f exists such that $H_N(f, I)$ is finite and so we have a reasonable problem, it is sufficient to impose the regularity assumption that $|E(X_n)| < \infty$ for all n.

3. Solution of the Single-Period Model

Intuitively, the solution to this problem is that f^* should be the smallest number such that $B_N^*(I + f) \geqq \beta$. We suspect this to be the case because (a) increasing the amount on hand at the start of the period should increase the average level of the pool during the period, and (b) increasing the amount on hand at the start of the period should decrease the probability that some demand for funds will not be met, so (c) f^* should be the smallest feasible value of f, where f is feasible if $I_0 = I + f$ satisfies (4). We shall not give a rigorous proof here because it is a special case of the more general proof in §5.

We require an explicit expression for $B_N^*(I + f)$. By assumption, $X_1, X_2, \cdots, X_N$ are independent random variables; let $X_n(x) = \Pr\{X_n \leqq x\}$. Now define $g_n = d_n + x_n$, so that $G_n(x) = \Pr\{g_n \leqq x\} = X_n(x - d_n)$, $n \in N'$. Let

$$B_n(x; I + f) = \Pr\{I_n \leqq x, I_{n-1} \geqq 0, \cdots, I_1 \geqq 0 \mid I_0 = I + f\},$$

i.e., $B_n(1x; I + f)$ is the joint probability that with an initial inventory of $I + f$, the inventory level on day n is $\leqq x$ and the inventory level has never been negative. Clearly,

$$B_0(x; I) = 1(I + f), \qquad -\infty < x < \infty, \qquad (6)$$

where $1(t)$ is the unit step function at t. Rewriting (1) as

$$I_n = I_{n-1} + g_n, \qquad n = 1, \cdots, N, \qquad (1')$$

we see that, for $n = 1, 2, \cdots, N$,

$$B_n(x; I + f \mid g_n = y) = B_{n-1}(x - y; I + f) - B_{n-1}(0^-; I + f) \quad \text{if} \quad y \leq x,$$
$$= 0 \quad \text{if} \quad y > x,$$

so

$$(7) \quad B_n(x; I + f) = \int_{-\infty}^{x} B_{n-1}(x - y; I + f)\, dG_n(y) - B_{n-1}(0^-; I + f)G_n(x)$$

for $n \in N'$ and $-\infty < x < \infty$. Together, (6) and (7) can be used to recursively calculate $B_n(x; I + f)$ for all values of $n \in N'$. Now observe that

$$(8) \quad \begin{aligned} B_N^*(I + f) &= \Pr\{E_1 = 0, \cdots, E_N = 0 \mid I_0 = I + f\} \\ &= \Pr\{I_n \geqq 0, \cdots, I_1 \geqq 0 \mid I_0 = I + f\} \\ &= B_N(\infty; I + f) - B_N(0^-; I + f). \end{aligned}$$

To describe the cash balance on day n we let $P_n(x; I + f) = \Pr\{Pn \leqq x \mid I_0 = I + f\}$; obviously

$$(9) \quad P_0(x; I + f) = 1(I + f)$$

and from (2) we obtain

$$(10) \quad \begin{aligned} P_n(x; I + f) &= \int_{0-}^{\infty} G_n(x - y)\, dP_{n-1}(y; I + f) \\ &= \int_{0-}^{\infty} P_{n-1}(y; I + f)\, dX_n(x - d_n - y), \end{aligned}$$

for $0 \leqq x < \infty$ and $n \in N'$. Together (9) and (10) provide a recursive relation to find $P_n(x; I + f)$ for all $n \in N'$.

To describe the nature of the emergency loans, it is a simple matter to use (3) to obtain

$$(11) \quad E_n(x; I + f) \triangleq \Pr\{E_n \leqq x \mid I_0 = I + f\} = 1 - P_n(-x; I + f)$$

for $x \geqq 0$ and each $n \in N'$.

4. The Multi-Period Model and Its Solution

Suppose the planning interval is divided into M periods containing $N_1, N_2, \cdots, N_M$ days each. Each of these periods will be called a month, and it will make no essential difference if we assume that each month has the same number of days, denoted by N. Time zero will be the beginning of the first day of the first month, and the pool level at the start of period m is I_m.

We are to select the amount of money, f_m, that will be instantaneously added to the pool at the start of period m, $m = 1, \cdots, M$. For the moment we will think of f_m as the money raised by a bond issue at the start of period m. Let $\alpha_{m+r}f_m$ be the amount of money paid out in period $m + r$ in return for the bond issue of period m, where $r = 1, \cdots, M - m$. For example, suppose the planning period is January 1, 1969 to January 31, 1970 so $M = 13$, f_1 is a \$100 million bond issue at 6 per cent per year for 40 years, where the interest is paid on July 1 and January 1 of each year. Then there would be interest payments of 3 million dollars on July 1, 1969 and January 1, 1970 in partial repayment of the 100 million dollars raised on January 1, 1969. Thus, $\alpha_6 = \alpha_{12} = 0.03$ and all other values of α_{1+r} are zero. The essential point is: α_{m+r} is the average amount of money to leave the pool during the $(m + r)$th month due to increasing the level of the pool by one dollar on the first day of the mth month, and it leaves on the first day of the $(m + r)$th month.

A similar notation is used to represent the repayment of an emergency loan. We assume that these loans are repaid at the end of the month in which they occur, and the amount repaid for a loan on day n of size E_n is $\eta_n E_n$.

The problem is to choose $f_1, \cdots, f_M$ to minimize the average level of the pool subject to the constraint that within each period the probability that no emergency loans are made must be at least some given number (less than one).

In the multiperiod case it is important to specify when each value of f_m is to be chosen. If f_m is selected at the start of period m, then we just have a sequence of one-period problems which we already know how to solve. The interesting problem is when all the f_m are chosen at time zero. We shall present the details of the model for $M = 2$ only, to simplify the presentation; it will be apparent how to extend the model for any finite value of M.

The average cash balance during the first two periods, $H(f_1, f_2; I)$ is given by

$$H(f_1, f_2; I) = \frac{1}{N} \sum_{n=1}^{N} E(P_n; I + f_1) + \frac{1}{N} \int_0^\infty \sum_{n=1}^{N} E(P_n; x + f_2)\, dP_N(x; I + f_1). \tag{12}$$

Only the second term in (12) requires explanation at the end of day N, the level of the pool is the nonnegative random variable P_N which has d.f. $P_N(x; I + f_1)$ since $P_0 = I + f_1$; using the fact that the pool process is a Markov process we find that

$$E(P_{N+n}; P_0) = \int_0^x E(P_{N+n} \mid P_N = x)\, dP_N(x; I + f_1).$$

The constraints of the problem can be neatly written if we generalize the definition of the function $B_N^*(\cdot)$. Hereafter, let

$$B_N^*(y) = \Pr\{E_{k+1} = 0, \cdots, E_{k+N} = 0 \mid P_k = y\};$$

the constraints of the problem are written as

$$B_N^*(I_0 + f_1) \geqq \beta_1, \tag{13a}$$

$$\int_{0-}^\infty B_N^*(x + f_2)\, dP_N(x) \geqq \beta_2. \tag{13b}$$

Assuming that $|E(x_n)| < \infty$ to insure a meaningful problem, the smallest values of f_1 and f_2 that satisfy (13a) and (13b), denoted by f_1^* and f_2^* respectively, are optimal. This result is proven in the next section. By examining the proof, one sees that the optimal policy is not changed if we change the objective function to

$$H(f_1, f_2; I) = \frac{1}{N} \sum_{n=1}^{N} a_n E(P_n; I + f_1) + \frac{1}{N} \int_0^\infty \sum_{n=1}^{N} a_{N+n} E(P_n; x + f_2)\, dP_N(x; I + f_1) + c(f_1, f_2) \tag{12'}$$

where $a_n \geqq 0$, $n = 1, 2, \cdots, 2N$, and $c(f_1, f_2)$ is nondecreasing in both f_1 and f_2.

We can interpret a_n as a discount factor and $c(\cdot, \cdot)$ as the cost of acquiring f_i dollars at the start of period i, $i = 1, 2$. The monotonicity condition on $c(\cdot, \cdot)$ then means that the marginal cost of obtaining a dollar is always nonnegative.

Now let us introduce interest returns at rate σ for the money in the pool. Then (1) and (2) become

$$I_n = I_{n-1} + f_n + d_n + X_n + \sigma \max[0, I_{n-1}], \qquad n = 1, 2, \cdots,$$

$$P_n = \sigma \max[0, P_{n-1}] + f_n + d_n + X_n, \qquad n = 1, 2, \cdots,$$

and a moment's reflection should convince the reader that this modification of (1) and (2) will not change the structure of the problem.

5. Proof that f_1^* and f_2^* are Optimal

In this section we shall prove that the smallest values of f_1 and f_2 that satisfy (13a) and (13b) minimize (12). It will be apparent that this proof for a two-period problem is applicable to any problem with a finite number of periods.

It is intuitively obvious (and laborious to prove rigorously) that $B_N^*(I + f_1)$ is increasing in f_1 and independent of f_2, $P_N(x)$ is increasing in f_1, and $\int_{0-}^{\infty} B_N^*(x + f_2)\, dP_N(x)$ is increasing in both f_1 and f_2.

Minimizing $H(f_1, f_2, I)$ is equivalent to minimizing $\sum_{n=1}^{N} E(P_n; I + f_1) + \sum_{n=N+1}^{2N} E(P_n; I + f_1, f_2)$; since the first term above is independent of f_2 and increasing with f_1, it is minimized by the smallest value of f_1 satisfying (12), namely f_2^*. By definition we have

$$\text{(14a)} \quad P_{N+1} = \max[0, P_N] + d_{N+1} + X_{N+1} + \textstyle\sum_{n=1}^{N} \eta_n \min[0, P_n] - \alpha f_1 + f_2,$$

$$\text{(14b)} \quad P_n = \max[0, P_{n-1}] + d_n + X_n, \qquad n = N + 2, \cdots, 2N.$$

It is easy to show that P_N is stochastically increasing with f_2, and from (14b) we obtain that P_n is stochastically increasing with P_N and hence $E(P_n; I + f_1, f_2)$ increases with f_2, $n = N + 2, \cdots, 2N$. From (2) and the principle of mathematical induction we can show that $\Pr\{E_{N+1} = 0, \cdots, E_{2N} = 0 \mid P_{N+1} = y\}$ is increasing with y. From (14a) we see that P_{N+1} is the sum of a random variable Y and a deterministic quantity k, where

$$Y = \max[0, P_N] + d_{N+1} + X_{N+1} + \textstyle\sum_{n=1}^{N} \eta_n \min[0, P_n]$$

and $k = f_2 - \alpha_2 f_1$, and Y is stochastically increasing with f_1. We can write

$$\int_{0-}^{\infty} B_N(x + f_2)\, dP_N(t)$$

$$= \int_{0-}^{\infty} \Pr\{E_{N+1} = 0, \cdots, E_{2N} = 0 \mid P_{N+1} = x\}\, d\Pr\{Y = x - k\} \triangleq q(k),$$

and $q(k)$ is an increasing function of k.

Therefore, the smallest value of k satisfying $q(k) \geqq \beta_2$ gives the smallest feasible value of P_{N+1} and hence minimizes $\sum_{n=N+1}^{2N} E(P_n; I + f_1, f_2)$ subject to (13b). But this means that $f_2 = q^{-1}(\beta) + \alpha_1 f_1$ gives the optimal value of f_2 for *any* value of f_1, so if we set $f_1 = f_1^*$ and $f_2^* = q^{-1}(\beta) + \alpha_1 f_1^*$ the pair (f_1^*, f_2^*) must minimize (12) subject to (13a) and (13b), concluding our demonstration. This is the same type of optimal policy that is given in [3].

6. Discussion of Results

In the preceding sections we have given an exact mathematical solution to a mathematical formulation of a cash balance problem. Since the model is only an approximation of the actual problem (presumably it is a good approximation), we need not restrict our attention to the exact answer.

The exact answer found above can be phrased, "never have any more cash on hand than is required", and it seems self-evident. To obtain exact results, one is required to evaluate the function $B_N^*(\cdot)$; this may be a difficult numerical problem, especially since we are interested in the upper tail of the functions. There is also a difficult estimation problem, since the tails of the d.f.'s $X_n(\cdot)$ will be hard to estimate.

However, these difficulties can be overcome rather easily if we restrict our attention to "ball park" estimates of f_n^*. Well-known probabilistic results (e.g., Chebychev's inequality and extreme value theory) can be used to approximate $B_N^*(\cdot)$, and order of magnitude estimates of f_n^* can be obtained. This procedure may preclude making a thorough analysis when the preliminary results are striking.

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